

# Bounds on Turán determinants

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## Abstract

Let  $\mu$  denote a symmetric probability measure on  $[-1, 1]$  and let  $(p_n)$  be the corresponding orthogonal polynomials normalized such that  $p_n(1) = 1$ . We prove that the normalized Turán determinant  $\Delta_n(x)/(1-x^2)$ , where  $\Delta_n = p_n^2 - p_{n-1}p_{n+1}$ , is a Turán determinant of order  $n-1$  for orthogonal polynomials with respect to  $(1-x^2)d\mu(x)$ . We use this to prove lower and upper bounds for the normalized Turán determinant in the interval  $-1 < x < 1$ .

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## 1 Introduction

In the following we will deal with polynomial sequences  $(p_n)$  satisfying

$$\begin{aligned} xp_n(x) &= \gamma_n p_{n+1}(x) + \alpha_n p_{n-1}(x), \quad n \geq 0, \\ \alpha_n + \gamma_n &= 1, \quad \alpha_n > 0, \quad \gamma_n > 0, \quad n \geq 1, \\ p_0(x) &= 1, \quad \alpha_0 = 0, \quad 0 < \gamma_0 \leq 1. \end{aligned} \tag{1}$$

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Note that  $(p_n)$  is uniquely determined by (1) from the recurrence coefficients  $\alpha_n, \gamma_n$ . It is well-known that the polynomials  $p_n$  are orthogonal with respect to a symmetric probability measure  $\mu$  with compact support.

Define the Turán determinant by

$$\Delta_n(x) = p_n^2(x) - p_{n-1}(x)p_{n+1}(x), \quad n \geq 1. \quad (2)$$

In [11] the second author proved non-negativity of the Turán determinant (2) under certain monotonicity conditions on the recurrence coefficients, thereby obtaining results for new classes of polynomials and unifying old results.

If  $\gamma_0 = 1$  the polynomials satisfy  $p_n(1) = 1$  and therefore the *normalized* Turán determinant  $\Delta_n(x)/(1 - x^2)$  is a polynomial in  $x$ .

We shall prove estimates of the form

$$c\Delta_n(0) \leq \frac{\Delta_n(x)}{1 - x^2} \leq C\Delta_n(0), \quad -1 < x < 1, \quad (3)$$

under certain regularity conditions on the recurrence coefficients. We prove e.g. an inequality of the left-hand type if  $(\alpha_n)$  is increasing and concave, see Theorem 2.5. In Theorem 2.7 we give an inequality of the right-hand type.

Our results depend on a simple relation between the Turán determinants of order  $n$  and  $n - 1$  (Proposition 2.1) and the following observation: The normalized Turán determinant is essentially a Turán determinant of order  $n - 1$  for the polynomials  $(q_n)$  defined by (15) below, and if  $\mu$  is the orthogonality measure of  $(p_n)$ , then  $(q_n)$  are orthogonal with respect to the measure  $(1 - x^2)d\mu(x)$ . See Theorem 2.3 and Remark 2.4 for a precise statement.

In Proposition 2.11 we prove non-negativity of the Turán determinant for the normalized polynomials  $q_n(x)/q_n(1)$  provided the sequence  $(\alpha_n)$  is increasing and concave (or under the weaker condition (19)).

Our work is motivated by results about ultraspherical polynomials, which we describe next.

For  $\alpha > -1$  let  $R_n^{(\alpha, \alpha)}(x) = P_n^{(\alpha, \alpha)}(x)/P_n^{(\alpha, \alpha)}(1)$  denote the symmetric Jacobi polynomials normalized to be 1 for  $x = 1$ , i.e.

$$R_n^{(\alpha, \alpha)}(x) = \frac{(-1)^n}{2^n(\alpha + 1)_n} (1 - x^2)^{-\alpha} \frac{d^n}{dx^n} (1 - x^2)^{n+\alpha}, \quad (4)$$

cf. [10]. We use the Pochhammer symbol  $(a)_n = a(a+1)\cdots(a+n-1)$ . The polynomials are orthogonal with respect to the symmetric weight function

$c_\alpha(1-x^2)^\alpha$  on  $] -1, 1[$ . Here  $1/c_\alpha = B(\alpha+1, 1/2)$ , so the weight is a probability density. We have  $R_n^{(\alpha, \alpha)}(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$  with  $\alpha = \lambda - \frac{1}{2}$ , where  $(P_n^{(\lambda)})$  are the ultraspherical polynomials in the notation of [10].

The corresponding Turán determinant of order  $n$

$$\Delta_n^{(\alpha)}(x) = R_n^{(\alpha, \alpha)}(x)^2 - R_{n-1}^{(\alpha, \alpha)}(x)R_{n+1}^{(\alpha, \alpha)}(x), \quad (5)$$

is clearly a polynomial of degree  $n$  in  $x^2$  and divisible by  $1-x^2$  since it vanishes for  $x = \pm 1$ . The following Theorem was proved in [12, pp. 381-382] and in [14, sect. 6]:

**Theorem 1.1.** *The normalized Turán determinant*

$$f_n^{(\alpha)}(x) := \Delta_n^{(\alpha)}(x)/(1-x^2) \quad (6)$$

is

- (i) strictly increasing for  $0 \leq x < \infty$  when  $\alpha > -1/2$ .
- (ii) constant equal to 1 for  $x \in \mathbb{R}$  when  $\alpha = -1/2$ .
- (iii) strictly decreasing for  $0 \leq x < \infty$  when  $-1 < \alpha < -1/2$ .

It is easy to evaluate  $f_n^{(\alpha)}$  at  $x = 0, 1$  giving

$$f_n^{(\alpha)}(0) = \mu_{[n/2]}^{(\alpha)} \mu_{[(n+1)/2]}^{(\alpha)}, \quad f_n^{(\alpha)}(1) = \frac{1}{2\alpha+2}, \quad (7)$$

where we have used the notation from [1]

$$\mu_n^{(\alpha)} = \frac{\mu_n}{\binom{n+\alpha}{n}}, \quad (8)$$

and  $\mu_n$  is the normalized binomial mid-coefficient

$$\mu_n = 2^{-2n} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot (2n)}. \quad (9)$$

**Corollary 1.2.** *For  $-1 < x < 1$  we have*

$$f_n^{(\alpha)}(0)(1-x^2) < \Delta_n^{(\alpha)}(x) < f_n^{(\alpha)}(1)(1-x^2) \text{ for } \alpha > -1/2 \quad (10)$$

while the inequalities are reversed when  $-1 < \alpha < -1/2$ . (For  $\alpha = -1/2$  all three terms are equal to  $1-x^2$ .)

For  $\alpha = 0$  the inequalities (10) reduce to  $(-1 < x < 1)$

$$\mu_{[n/2]}\mu_{[(n+1)/2]}(1-x^2) < P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) < \frac{1}{2}(1-x^2) \quad (11)$$

for Legendre polynomials  $(P_n)$ . This result was recently published in [1] using a SumCracker Package by Manuel Kauers, and it was conjectured that the monotonicity result remains true for ultraspherical polynomials when  $\alpha \geq -1/2$ . Clearly the authors have not been aware of the early results above.<sup>1</sup> Turán [13] proved that  $\Delta_n^{(0)}(x) > 0$  for  $-1 < x < 1$ . The proof in [12] of Theorem 1.1 is based on a formula relating the Turán determinant

$$\Delta_{n,\lambda}(x) = F_{n,\lambda}^2(x) - F_{n-1,\lambda}(x)F_{n+1,\lambda}(x)$$

of the normalized ultraspherical polynomials  $F_{n,\lambda}(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$  and the expression

$$D_{n,\lambda}(x) = \left[\frac{d}{dx}P_n^{(\lambda)}(x)\right]^2 - \frac{d}{dx}P_{n-1}^{(\lambda)}(x)\frac{d}{dx}P_{n+1}^{(\lambda)}(x),$$

namely (see [12, (5.9)])

$$\frac{\Delta_{n,\lambda}(x)}{1-x^2} = \frac{D_{n,\lambda}(x)}{n(n+2\lambda)[P_n^{(\lambda)}(1)]^2}. \quad (12)$$

See also [3]. Using the well-known formula for differentiation of ultraspherical polynomials

$$\frac{d}{dx}P_n^{(\lambda)}(x) = 2\lambda P_{n-1}^{(\lambda+1)}(x),$$

we see that

$$D_{n,\lambda}(x) = (2\lambda)^2 \left( [P_{n-1}^{(\lambda+1)}(x)]^2 - P_{n-2}^{(\lambda+1)}(x)P_n^{(\lambda+1)}(x) \right). \quad (13)$$

Except for the factor  $(2\lambda)^2$  this is the Turán determinant of order  $n-1$  for the ultraspherical polynomials corresponding to the parameter  $\lambda+1$ .

We see that this result is generalized in Theorem 2.3.

Since the proof of the monotonicity in Theorem 1.1 depends on the fact that the ultraspherical polynomials satisfy a differential equation, there is

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<sup>1</sup>Motivated by this conjecture the present authors found a proof of Theorem 1.1 close to the old proofs. During the preparation of the paper we found the references [12], [14].

little hope of extending the result to classes of orthogonal polynomials which do not satisfy a differential equation. We have instead attempted to find bounds for normalized Turán determinants without using monotonicity in the variable  $x$ .

This has also lead us to consider the following lower boundedness condition for general orthonormal polynomials  $(P_n)$ :

$$(LB) \quad \inf\{P_{n-1}^2(x) + P_n^2(x) \mid x \in \mathbb{R}, n \in \mathbb{N}\} > 0. \quad (14)$$

If the condition (LB) holds then necessarily  $\sum_{n=0}^{\infty} P_n^2(x) = \infty$  for all  $x \in \mathbb{R}$ . Therefore the orthogonality measure  $\mu$  is uniquely determined and has no mass points.

In Proposition 3.1 we prove that (LB) holds for symmetric orthonormal polynomials if the recurrence coefficients are increasing and bounded. It turns out that for the orthonormal symmetric Jacobi polynomials the condition (LB) holds if and only if  $\alpha \geq 1/2$ .

The theory is applied to continuous  $q$ -ultraspherical polynomials in Section 4.

Concerning the general theory of orthogonal polynomials we refer the reader to [10],[9],[6].

## 2 Main results

**Proposition 2.1.** *In addition to (1) assume that  $\alpha_n \neq \gamma_n$  for  $n = 1, 2, \dots$ . Then for  $n \geq 2$  there holds*

$$\Delta_n = \frac{(\gamma_n - \alpha_n)\alpha_{n-1}}{(\gamma_{n-1} - \alpha_{n-1})\gamma_n} \Delta_{n-1} + \frac{\alpha_n - \alpha_{n-1}}{(\gamma_{n-1} - \alpha_{n-1})\gamma_n} (p_{n-1}^2 + p_n^2 - 2xp_{n-1}p_n).$$

*Proof.* By the recurrence relation we can remove either  $p_{n+1}$  or  $p_{n-1}$  from the formula defining  $\Delta_n$ . In this way we obtain two equalities

$$\begin{aligned} \gamma_n \Delta_n &= \alpha_n p_{n-1}^2 + \gamma_n p_n^2 - xp_{n-1}p_n, \\ \alpha_n \Delta_n &= \alpha_n p_n^2 + \gamma_n p_{n+1}^2 - xp_n p_{n+1}. \end{aligned}$$

We replace  $n$  by  $n-1$  in the second equality and multiply both sides by  $\gamma_n - \alpha_n$ . Next we subtract the resulting equality from the first one multiplied by  $\gamma_{n-1} - \alpha_{n-1}$ . In this way we obtain after obvious simplifications

$$\begin{aligned} &(\gamma_{n-1} - \alpha_{n-1})\gamma_n \Delta_n - (\gamma_n - \alpha_n)\alpha_{n-1} \Delta_{n-1} \\ &= (\alpha_n \gamma_{n-1} - \alpha_{n-1} \gamma_n)(p_{n-1}^2 + p_n^2) - (\gamma_{n-1} - \gamma_n - \alpha_{n-1} + \alpha_n) xp_{n-1}p_n. \end{aligned}$$

Taking into account that  $\alpha_k + \gamma_k = 1$  for  $k \geq 1$  gives

$$(\gamma_{n-1} - \alpha_{n-1})\gamma_n\Delta_n - (\gamma_n - \alpha_n)\alpha_{n-1}\Delta_{n-1} = (\alpha_n - \alpha_{n-1})(p_{n-1}^2 + p_n^2 - 2xp_{n-1}p_n).$$

□

Proposition 2.1 implies

**Corollary 2.2.** [11, Thm. 1] *In addition to (1) assume that one of the following conditions holds.*

- (i)  $(\alpha_n)$  is increasing and  $\alpha_n \leq \gamma_n$ ,  $n \geq 1$ .
- (ii)  $(\alpha_n)$  is decreasing and  $\alpha_n \geq \gamma_n$ ,  $n \geq 1$ . Furthermore, assume that  $\gamma_0 = 1$  or  $\gamma_0 \leq \gamma_1/(1 - \gamma_1)$ .

Then

$$\Delta_n(x) > 0$$

for  $-1 < x < 1$ .

*Proof.* Assume first the additional condition  $\alpha_n \neq \gamma_n$  for all  $n \geq 0$ . Since  $p_{n-1}^2 + p_n^2 - 2xp_{n-1}p_n > 0$  for  $-1 < x < 1$ , it suffices in view of Proposition 2.1 to show that  $\Delta_1 > 0$ . We have

$$\gamma_1\Delta_1(x) = \alpha_1p_0^2 + \gamma_1p_1^2 - xp_0p_1 = \frac{\alpha_1\gamma_0^2 + (\gamma_1 - \gamma_0)x^2}{\gamma_0^2},$$

hence  $\Delta_1 > 0$  if  $\gamma_1 \geq \gamma_0$ . If  $\gamma_1 < \gamma_0$ , we have

$$\gamma_1\Delta_1(x) > \gamma_1\Delta_1(1) = \frac{\alpha_1(1 - \gamma_0)(\gamma_1/\alpha_1 - \gamma_0)}{\gamma_0^2}.$$

This is clearly non-negative in case (i) because  $\gamma_1/\alpha_1 \geq 1$ , but also in case (ii) because of the assumptions on  $\gamma_0$ .

Assume next in case (i) that there is an  $n$  such that  $\alpha_n = \gamma_n$  and let  $n_0 \geq 1$  be the smallest  $n$  with this property. Denoting  $\alpha = \lim \alpha_n$ , then clearly  $\alpha_n \leq \alpha \leq 1 - \alpha \leq \gamma_n$  for all  $n$  and hence  $\alpha_n = \gamma_n = 1/2$  for  $n \geq n_0$ . Therefore

$$\Delta_n = p_{n-1}^2 + p_n^2 - 2xp_{n-1}p_n > 0$$

for  $n \geq n_0$ ,  $-1 < x < 1$ . The formula of Proposition 2.1 can be applied for  $2 \leq n < n_0$  and the proof of the first case carries over. Equality in case (ii) is treated similarly. □

From now on we will assume that additionally  $\gamma_0 = 1$ . In this way the polynomials  $p_n$  are normalized at  $x = 1$  so that  $p_n(1) = 1$ . It follows by induction that  $p_n$  has all its zeros in  $] -1, 1[$ , hence that the support of the orthogonality measure  $\mu$  for  $(p_n)$  is contained in  $[-1, 1]$ . Since  $p_n(-x) = (-1)^n p_n(x)$  we conclude that  $p_n(-1) = (-1)^n$ . Therefore for any  $n \geq 0$  the polynomial  $p_{n+2} - p_n$  is divisible by  $x^2 - 1$ . Defining

$$q_n(x) = \frac{p_{n+2}(x) - p_n(x)}{x^2 - 1}, \quad n \geq 0, \quad (15)$$

$q_n(x)$  is a polynomial of degree  $n$ . Moreover, an easy calculation shows that the polynomials  $q_n$  are orthogonal with respect to the probability measure  $d\nu(x) = \frac{1}{\gamma_1}(1 - x^2)d\mu(x)$ . By the recurrence relation (1) and by  $\gamma_0 = 1$  we obtain that the polynomials  $q_n$  satisfy

$$xq_n(x) = \gamma_{n+2}q_{n+1}(x) + \alpha_n q_{n-1}(x), \quad n \geq 0, \quad q_0 = 1/\gamma_1. \quad (16)$$

The following theorem contains a fundamental formula relating the Turán determinants of the polynomials  $p_n$  and  $q_n$ .

**Theorem 2.3.** *For  $n \geq 1$  we have*

$$\frac{\Delta_n(x)}{1 - x^2} = \alpha_n \gamma_n q_{n-1}^2(x) - \alpha_{n-1} \gamma_{n+1} q_{n-2}(x) q_n(x). \quad (17)$$

*Proof.* By (1) we get

$$\begin{aligned} p_{k+1} - xp_k &= \alpha_k(p_{k+1} - p_{k-1}) = \alpha_k(x^2 - 1)q_{k-1}, \\ xp_k - p_{k-1} &= \gamma_k(p_{k+1} - p_{k-1}) = \gamma_k(x^2 - 1)q_{k-1}. \end{aligned}$$

Therefore

$$\begin{aligned} (x^2 - 1)^2 [\alpha_n \gamma_n q_{n-1}^2(x) - \alpha_{n-1} \gamma_{n+1} q_{n-2}(x) q_n(x)] \\ = (p_{n+1} - xp_n)(xp_n - p_{n-1}) - (p_n - xp_{n-1})(xp_{n+1} - p_n) \\ = (1 - x^2)(p_n^2 - p_{n-1}p_{n+1}). \end{aligned}$$

□

**Remark 2.4.** If we define  $\tilde{q}_0 = \gamma_1 q_0 = 1$  and

$$\tilde{q}_n = \frac{\gamma_1 \cdots \gamma_{n+1}}{\alpha_1 \cdots \alpha_n} q_n, \quad n \geq 1$$

we have

$$\frac{\Delta_n(x)}{1-x^2} = \frac{\gamma_n}{\alpha_n} \left( \frac{\alpha_1 \cdots \alpha_n}{\gamma_1 \cdots \gamma_n} \right)^2 [\tilde{q}_{n-1}^2(x) - \tilde{q}_{n-2}(x)\tilde{q}_n(x)], \quad (18)$$

showing that the normalized Turán determinant (17) is proportional to a Turán determinant of order  $n-1$  of the renormalized polynomials  $(\tilde{q}_n)$ . They satisfy the recursion equation ( $\tilde{q}_{-1} := 0$ )

$$x\tilde{q}_n = \alpha_{n+1}\tilde{q}_{n+1} + \gamma_{n+1}\tilde{q}_{n-1}, \quad n \geq 0.$$

**Theorem 2.5.** *Assume that  $(p_n)$  satisfies (1) with  $\gamma_0 = 1$ . Let  $(\alpha_n)$  be increasing,  $\alpha_n \leq 1/2$  and*

$$\alpha_n - \alpha_{n-1} \geq \frac{\alpha_n}{1 - \alpha_n}(\alpha_{n+1} - \alpha_n), \quad n \geq 1. \quad (19)$$

(e.g. (19) is satisfied if  $\alpha_n$  is concave). Then for  $\Delta_n(x)$  defined by (2) we have

$$\frac{\Delta_n(x)}{1-x^2} \geq c\Delta_n(0), \quad -1 < x < 1, \quad n \geq 1,$$

where  $c = 2\alpha_1\gamma_2/\gamma_1$ .

*Proof.* Observe that (19) is equivalent to  $(\alpha_n\gamma_{n+1})$  being increasing. Let

$$D_n(x) = \gamma_n q_{n-1}^2(x) - \gamma_{n+1} q_{n-2}(x)q_n(x).$$

Since  $\alpha_n \geq \alpha_{n-1}$  Theorem 2.3 implies that

$$\frac{\Delta_n(x)}{1-x^2} \geq \alpha_{n-1} D_n(x). \quad (20)$$

By (16) we can remove  $q_n$  or  $q_{n-2}$  from the expression defining  $D_n$ . In this way we obtain

$$\begin{aligned} D_n &= \alpha_{n-1} q_{n-2}^2 + \gamma_n q_{n-1}^2 - x q_{n-2} q_{n-1}, \\ \frac{\alpha_{n-1}}{\gamma_{n+1}} D_n &= \frac{\alpha_{n-1} \gamma_n}{\gamma_{n+1}} q_{n-1}^2 + \gamma_{n+1} q_n^2 - x q_{n-1} q_n. \end{aligned} \quad (21)$$

Replacing  $n$  by  $n-1$  in the second equality and subtracting it from the first we find

$$D_n - \frac{\alpha_{n-2}}{\gamma_n} D_{n-1} = \frac{\alpha_{n-1} \gamma_n - \alpha_{n-2} \gamma_{n-1}}{\gamma_n} q_{n-2}^2 \geq 0.$$



By iterating the inequality between  $D_n$  and  $D_{n-1}$  we obtain

$$D_n \geq \frac{\alpha_1 \cdots \alpha_{n-2}}{\gamma_3 \cdots \gamma_n} D_2. \quad (22)$$

From (21) we get

$$D_2 = \alpha_1 q_0^2 + \gamma_2 q_1^2 - x q_0 q_1 = \frac{\alpha_1}{\gamma_1^2} + \frac{\gamma_2 x^2}{\gamma_1^2 \gamma_2^2} - \frac{x^2}{\gamma_1^2 \gamma_2} = \frac{\alpha_1}{\gamma_1^2}, \quad (23)$$

so (20) implies

$$\frac{\Delta_n(x)}{1-x^2} \geq \frac{\alpha_1 \cdots \alpha_n}{\gamma_1 \cdots \gamma_n} \frac{\alpha_1 \gamma_2}{\alpha_n \gamma_1} \geq \frac{\alpha_1 \cdots \alpha_n}{\gamma_1 \cdots \gamma_n} \frac{2\alpha_1 \gamma_2}{\gamma_1}.$$

The conclusion follows from the next lemma.  $\square$

**Lemma 2.6.** *Under the assumptions of Theorem 2.5 there holds*

$$\Delta_n(0) \leq \frac{\alpha_1 \cdots \alpha_n}{\gamma_1 \cdots \gamma_n} \leq \frac{\gamma_1}{\alpha_1} \Delta_n(0), \quad n \geq 1.$$

*Proof.* Denote

$$h_n = \frac{\gamma_1 \cdots \gamma_n}{\alpha_1 \cdots \alpha_n}.$$

By (1) we have

$$p_{2n}(0) = (-1)^n \frac{\alpha_1 \alpha_3 \cdots \alpha_{2n-1}}{\gamma_1 \gamma_3 \cdots \gamma_{2n-1}}.$$

Hence

$$\Delta_{2n}(0) h_{2n} = p_{2n}^2(0) h_{2n} = \prod_{k=1}^n \frac{\alpha_{2k-1}}{\alpha_{2k}} \prod_{k=1}^n \frac{\gamma_{2k}}{\gamma_{2k-1}} \leq 1.$$

On the other hand

$$\begin{aligned} \Delta_{2n+1}(0) h_{2n+1} &= -p_{2n}(0) p_{2n+2}(0) h_{2n+1} \\ &= \prod_{k=1}^n \frac{\alpha_{2k-1}}{\alpha_{2k}} \prod_{k=1}^n \frac{\gamma_{2k}}{\gamma_{2k-1}} = \Delta_{2n}(0) h_{2n} \leq 1. \end{aligned}$$

Moreover

$$\Delta_{2n}(0) h_{2n} = \prod_{k=1}^n \frac{\alpha_{2k-1}}{\alpha_{2k}} \prod_{k=1}^n \frac{\gamma_{2k}}{\gamma_{2k-1}} \geq \prod_{k=2}^{2n} \frac{\alpha_{k-1}}{\alpha_k} \prod_{k=2}^{2n} \frac{\gamma_k}{\gamma_{k-1}} = \frac{\alpha_1 \gamma_{2n}}{\gamma_1 \alpha_{2n}} \geq \frac{\alpha_1}{\gamma_1}.$$

$\square$

Theorem 2.5 has the following counterpart and the proof is very similar:

**Theorem 2.7.** *Assume that  $(p_n)$  satisfies (1) with  $\gamma_0 = 1$ . Let  $\alpha_n, n \geq 1$  be decreasing,  $\alpha_n \geq \frac{1}{2}$  and*

$$\alpha_n - \alpha_{n-1} \leq \frac{\alpha_n}{1 - \alpha_n}(\alpha_{n+1} - \alpha_n), \quad n \geq 2. \quad (24)$$

*Then for  $\Delta_n(x)$  defined by (2) we have*

$$\frac{\Delta_n(x)}{1 - x^2} \leq C \Delta_n(0), \quad -1 < x < 1, \quad n \geq 1,$$

*where  $C = 2\gamma_2$ . (Note that (24) implies convexity of  $\alpha_n, n \geq 1$ .)*

**Remark 2.8.** Note that the normalized symmetric Jacobi polynomials  $p_n(x) = R_n^{(\alpha, \alpha)}(x)$  given by (4) satisfy (1) with

$$\gamma_n = \frac{n + 2\alpha + 1}{2n + 2\alpha + 1}, \quad \alpha_n = \frac{n}{2n + 2\alpha + 1}. \quad (25)$$

(In the case of  $\alpha = -1/2$ , i.e. Chebyshev polynomials of the first kind, these formulas shall be interpreted as  $\gamma_0 = 1, \alpha_0 = 0$ .)

For  $\alpha \geq -1/2$  we have  $(\alpha_n)$  is increasing and concave and  $c = 1$ .

For  $-1 < \alpha \leq -1/2$  the sequence  $(\alpha_n)$  is decreasing, (24) holds and  $C = 1$ .

The statement about the constants  $c, C$  follows from Corollary 1.2. However we cannot expect  $c = 1$  in general, because it is easy to construct an example, where the normalized Turán determinant (17) is not monotone for  $0 < x < 1$ .

Consider the sequence  $(\alpha_n) = (0, 1/2 - 3\varepsilon, 1/2 - 2\varepsilon, 1/2 - \varepsilon, 1/2, 1/2, \dots)$ , which is increasing and concave for  $0 < \varepsilon < 1/8$ . In this case the Turán determinant  $\tilde{q}_2^2 - \tilde{q}_1\tilde{q}_3$  is proportional to  $f(x) = x^4 + A(\varepsilon)x^2 + B(\varepsilon)$ , where

$$A(\varepsilon) = 4\varepsilon^2 + 3\varepsilon - 1/2, \quad B(\varepsilon) = (1/2 - 3\varepsilon)^2(1/2 - \varepsilon)(1/2 + 2\varepsilon)^2/\varepsilon.$$

Clearly,  $f$  is not monotone for  $0 < x < 1$  when  $\varepsilon$  is small.

**Corollary 2.9.** *Under the assumptions of Theorem 2.5 and the additional hypothesis  $\lim \alpha_n = 1/2$ , the orthogonality measure  $\mu$  is absolutely continuous on  $(-1, 1)$  with a strictly positive and continuous density  $g(x) = d\mu(x)/dx$  satisfying*

$$g(x) \leq \frac{C}{\sqrt{1 - x^2}}.$$

*Proof.* The corresponding orthonormal polynomials  $(P_n)$  satisfy

$$xP_n = \lambda_n P_{n+1} + \lambda_{n-1} P_{n-1}, \quad (26)$$

where  $\lambda_n = \sqrt{\alpha_{n+1}\gamma_n}$ . We also have  $P_n = \delta_n p_n$ , where

$$\delta_n = \sqrt{\frac{\gamma_0 \cdots \gamma_{n-1}}{\alpha_1 \cdots \alpha_n}}, \quad n \geq 1, \quad \delta_0 = 1,$$

and  $\lim \lambda_n = 1/2$ . Since

$$\lambda_{n+1} - \lambda_n = \frac{\alpha_{n+2}(\gamma_{n+1} - \gamma_n) + \gamma_n(\alpha_{n+2} - \alpha_{n+1})}{\sqrt{\alpha_{n+2}\gamma_{n+1}} + \sqrt{\alpha_{n+1}\gamma_n}},$$

the monotonicity of  $(\alpha_n), (\gamma_n)$  implies that

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

By the theorem in [8] we conclude that the orthogonality measure  $\mu$  has a positive continuous density  $g(x)$  for  $-1 < x < 1$ . Furthermore, it is known from this theorem that

$$\lim_{n \rightarrow \infty} [P_n^2(x) - P_{n-1}(x)P_{n+1}(x)] = \frac{2\sqrt{1-x^2}}{\pi g(x)},$$

uniformly on compact subsets of  $] -1, 1[$ . For another proof of this result see [5, p. 201], where it is also proved that  $(P_n(x))$  is uniformly bounded on compact subsets of  $] -1, 1[$  for  $n \rightarrow \infty$ . We have

$$\Delta_n(x) = \frac{1}{\delta_n^2} (P_n^2(x) - k_n P_{n-1}(x)P_{n+1}(x)),$$

where

$$k_n = \frac{\delta_n^2}{\delta_{n-1}\delta_{n+1}} = \sqrt{\frac{\alpha_{n+1}\gamma_{n-1}}{\alpha_n\gamma_n}},$$

and it follows that  $\lim k_n = 1$ . Using that

$$\frac{\Delta_n(x)}{\Delta_n(0)} = \frac{P_n^2(x) - k_n P_{n-1}(x)P_{n+1}(x)}{P_n^2(0) - k_n P_{n-1}(0)P_{n+1}(0)},$$

we get the result. □

In analogy with the proof of Corollary 2.9 we get

**Corollary 2.10.** *Under the assumptions of Theorem 2.7 and the additional hypothesis  $\lim \alpha_n = 1/2$ , the orthogonality measure  $\mu$  is absolutely continuous on  $(-1, 1)$  with a strictly positive and continuous density  $g(x) = d\mu(x)/dx$  satisfying*

$$g(x) \geq \frac{C}{\sqrt{1-x^2}}.$$

We now return to the polynomials  $(q_n)$  defined in (15) and prove that they have a non-negative Turán determinant after normalization to being 1 at 1. The polynomials  $q_n$  are orthogonal with respect to a measure supported by  $[-1, 1]$ . Therefore  $q_n(1) > 0$ .

**Proposition 2.11.** *Under the assumptions of Theorem 2.5 we have for  $n \geq 1$*

$$\frac{q_n^2(x)}{q_n^2(1)} - \frac{q_{n-1}(x)}{q_{n-1}(1)} \frac{q_{n+1}(x)}{q_{n+1}(1)} \geq 0.$$

*Proof.* Indeed, let  $Q_n(x) = \frac{q_n(x)}{q_n(1)}$ . Then

$$xQ_n = c_n Q_{n+1} + (1 - c_n)Q_{n-1},$$

where

$$c_n = \gamma_{n+2} \frac{q_{n+1}(1)}{q_n(1)}.$$

We will show that  $c_n$  is decreasing and  $c_n \geq 1/2$ . Then the conclusion follows from Corollary 2.2. But  $c_{n-1} \geq c_n$  is equivalent to

$$D_{n+1}(1) = \gamma_{n+1} q_n^2(1) - \gamma_{n+2} q_{n-1}(1) q_{n+1}(1) \geq 0,$$

which follows from (22) and (23). We will show that  $c_n \geq 1/2$  by induction. We have

$$c_0 = \gamma_2 \frac{q_1(1)}{q_0(1)} = 1.$$

Assume  $c_{n-1} \geq 1/2$ . By (19) the sequence  $(\alpha_n \gamma_{n+1})$  is increasing. Putting  $\alpha = \lim \alpha_n$  we then get

$$\alpha_n \gamma_{n+1} \leq \alpha(1 - \alpha) \leq \frac{1}{4}.$$

Using this and (16) leads to

$$1 = c_n + \frac{\alpha_n \gamma_{n+1}}{c_{n-1}} \leq c_n + \frac{1}{4c_{n-1}} \leq c_n + \frac{1}{2},$$

hence  $c_n \geq 1/2$ . □

### 3 Lower bound estimates

It turns out that Turán determinants can be used to obtain lower bound estimates for orthonormal polynomials. Recall that if the polynomials  $p_n$  satisfy the recurrence relation (1), then their orthonormal version  $(P_n)$  satisfy

$$xP_n = \lambda_n P_{n+1} + \lambda_{n-1} P_{n-1},$$

where  $\lambda_n = \sqrt{\alpha_{n+1}\gamma_n}$ .

**Proposition 3.1.** *Assume that the polynomials  $(P_n(x))$  satisfy*

$$xP_n = \lambda_n P_{n+1} + \lambda_{n-1} P_{n-1}, \quad n \geq 0, \quad (27)$$

*with  $P_{-1} = \lambda_{-1} = 0$ ,  $\lambda_n > 0$ ,  $n \geq 0$ , and  $P_0 = 1$ . If the sequence  $(\lambda_n)$  is increasing and  $\lim \lambda_n = L < \infty$ , then the (LB) condition (14) holds, viz.*

$$P_n^2(x) + P_{n-1}^2(x) \geq \frac{\lambda_0^2}{2L^2}.$$

*Proof.* This proof is inspired by [2, Thm. 3]. By replacing the polynomials  $P_n(x)$  by  $P_n(2Lx)$  we can assume that  $\lim \lambda_n = 1/2$ . This assumption implies that the corresponding Jacobi matrix is a contraction, because it can be majorized by the Jacobi matrix with entries  $\lambda_n = \frac{1}{2}$ . Therefore the orthogonality measure is supported by the interval  $[-1, 1]$ . In this way it suffices to consider  $x$  from  $[-1, 1]$  because the functions  $P_n^2(x)$  are increasing on  $[1, +\infty[$  and  $P_n^2(-x) = P_n^2(x)$ . Let

$$\mathcal{D}_n(x) = \lambda_{n-1} P_n^2(x) - \lambda_n P_{n-1}(x) P_{n+1}(x), \quad n \geq 1.$$

By (27) we can remove  $P_{n+1}$  to get

$$\mathcal{D}_n = \lambda_{n-1} P_{n-1}^2 + \lambda_{n-1} P_n^2 - x P_{n-1} P_n. \quad (28)$$

Alternatively we can remove  $P_{n-1}$  and obtain

$$\frac{\lambda_{n-1}}{\lambda_n} \mathcal{D}_n = \lambda_n P_{n+1}^2 + \frac{\lambda_{n-1}^2}{\lambda_n} P_n^2 - x P_n P_{n+1}. \quad (29)$$

Replacing  $n$  by  $n-1$  in (29) and subtracting it from (28) gives

$$\mathcal{D}_n - \frac{\lambda_{n-2}}{\lambda_{n-1}} \mathcal{D}_{n-1} = \frac{\lambda_{n-1}^2 - \lambda_{n-2}^2}{\lambda_{n-1}} P_{n-1}^2 \geq 0. \quad (30)$$

By iterating the inequality  $\mathcal{D}_n \geq (\lambda_{n-2}/\lambda_{n-1}) \mathcal{D}_{n-1}$ , we obtain

$$\mathcal{D}_n \geq \frac{\lambda_0}{\lambda_{n-1}} \mathcal{D}_1 = \frac{\lambda_0^2}{\lambda_{n-1}} \geq 2\lambda_0^2,$$

because by (28) we have  $\mathcal{D}_1 = \lambda_0$ . Now (28) implies for  $|x| \leq 1$

$$\mathcal{D}_n \leq \lambda_{n-1} P_{n-1}^2 + \lambda_{n-1} P_n^2 + \frac{1}{2} |x| (P_{n-1}^2 + P_n^2) \leq P_{n-1}^2 + P_n^2. \quad (31)$$

In the general case the lower bound is  $2(\lambda_0/(2L))^2$ .  $\square$

**Corollary 3.2.** *Under the assumptions of Proposition 3.1 with  $L = 1/2$  the orthogonality measure  $\mu$  is absolutely continuous with a continuous density  $g = d\mu(x)/dx$  on  $[-1, 1]$  satisfying*

$$g(x) \leq \frac{1}{2\pi\lambda_0^2} \sqrt{1-x^2}.$$

Furthermore,  $g(x) > 0$  for  $-1 < x < 1$ .

*Proof.* By assumptions the orthogonality measure is supported by  $[-1, 1]$ . By the proof of Proposition 3.1 we have

$$\mathcal{D}_n(x) \geq 2\lambda_0^2.$$

On the other hand by [8] and [5, p. 201] the orthogonality measure is absolutely continuous in the interval  $] -1, 1[$  with a strictly positive and continuous density  $g$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_{n-1}} \mathcal{D}_n(x) = \frac{2\sqrt{1-x^2}}{\pi g(x)},$$

uniformly on compact subsets of  $] -1, 1[$ , cf. the proof of Corollary 2.9. By Property (LB) there are no masses at  $\pm 1$ .  $\square$

The Jacobi polynomials  $P_n^{(\alpha, \alpha)}(x)$  in the standard notation of Szegő, cf. [10], are discussed in the Introduction. The corresponding orthonormal polynomials are denoted  $P_n(\alpha; x)$ . We recall that

$$c_\alpha \int_{-1}^1 [P_n^{(\alpha, \alpha)}(x)]^2 (1-x^2)^\alpha dx = \frac{2^{2\alpha+1} \Gamma(n+\alpha+1)^2}{(2n+2\alpha+1)n! \Gamma(n+2\alpha+1) B(\alpha+1, 1/2)}. \quad (32)$$

**Remark 3.3.** Corollary 3.2 is also obtained in [4, p.758].

**Corollary 3.4.** *Let  $(P_n(\alpha; x))$  denote the orthonormal symmetric Jacobi polynomials.*

(i) *For  $\alpha \geq 1/2$  we have*

$$\inf\{P_n^2(\alpha; x) + P_{n-1}^2(\alpha; x) \mid x \in \mathbb{R}, n \in \mathbb{N}\} \geq \frac{2}{2\alpha+3}.$$

(ii) *For  $-1 < \alpha < 1/2$  we have*

$$\inf\{P_n^2(\alpha; x) + P_{n-1}^2(\alpha; x) \mid x \in \mathbb{R}, n \in \mathbb{N}\} = 0.$$

*Proof.* Assume  $\alpha \geq 1/2$ . In this case we get from (25)

$$\lambda_n^2 = \frac{1}{4} \left[ 1 - \frac{4\alpha^2 - 1}{4(n+\alpha+1)^2 - 1} \right],$$

so  $(\lambda_n)$  is increasing with  $\lim \lambda_n = 1/2$ . By Proposition 3.1 we thus have

$$P_n^2 + P_{n-1}^2 \geq 2\lambda_0^2 = \frac{2}{2\alpha+3},$$

which shows (i).

In order to show (ii) we will make use of Hilb's asymptotic formula [10, Thm 8.21.12]:

$$\begin{aligned} \theta^{-1/2} \left( \sin \frac{\theta}{2} \right)^{\alpha+1/2} \left( \cos \frac{\theta}{2} \right)^{\alpha+1/2} P_n^{(\alpha, \alpha)}(\cos \theta) \\ = \frac{\Gamma(\alpha+n+1)}{n! \sqrt{2} N^\alpha} J_\alpha(N\theta) + O(n^{-3/2}), \end{aligned} \quad (33)$$

where  $\theta \in [c/n, \pi/2]$ ,  $N = n + \alpha + \frac{1}{2}$  and  $c > 0$  is fixed. Let  $j_\alpha$  denote the smallest positive zero of the Bessel function  $J_\alpha$ .

Defining  $\theta_n = j_\alpha/N$  we get

$$\begin{aligned} n^{-\alpha} P_n^{(\alpha, \alpha)}(\theta_n) &= O(n^{-3/2}), \\ n^{-\alpha} P_{n-1}^{(\alpha, \alpha)}(\theta_n) &= (1/\sqrt{2} + o(1)) J_\alpha(j_\alpha \frac{n + \alpha - 1/2}{n + \alpha + 1/2}) + O(n^{-3/2}) = O(n^{-1}). \end{aligned}$$

By (32) and Stirling's formula

$$c_\alpha \int_{-1}^1 [P_n^{(\alpha, \alpha)}(x)]^2 (1 - x^2)^\alpha dx \sim \frac{2^{2\alpha+1}}{B(\alpha + 1, 1/2)} n^{-1},$$

and hence

$$P_n^2(\alpha; \cos \theta_n) = O(n^{2\alpha-2}), \quad P_{n-1}^2(\alpha; \cos \theta_n) = O(n^{2\alpha-1}).$$

This shows that

$$P_n^2(\alpha; \cos \theta_n) + P_{n-1}^2(\alpha; \cos \theta_n) \rightarrow 0 \text{ when } \alpha < 1/2.$$

□

**Remark 3.5.** The example of symmetric Jacobi polynomials suggests that if  $\lambda_n$  is decreasing, then the condition of Corollary 3.4 (ii) may hold. This is not true because for  $\frac{1}{2} < \lambda_0 < \frac{1}{\sqrt{2}}$  and  $\lambda_n = \frac{1}{2}$  for  $n \geq 1$  we have a decreasing sequence and the corresponding Jacobi matrix has norm 1 because this is so for the cases  $\lambda_0 = \frac{1}{2}$  and  $\lambda_0 = 1/\sqrt{2}$ , which correspond to the Chebyshev polynomials of the second and first kind respectively. Furthermore, for  $n \geq 2$  we have by (28) and (30)

$$\mathcal{D}_n = \lambda_{n-1} P_n^2 - \lambda_n P_{n-1} P_{n+1} = \mathcal{D}_2 = \frac{2}{\lambda_0^2} [\lambda_0^4 - (\lambda_0^2 - \frac{1}{4}) x^2]$$

and for  $-1 < x < 1$

$$\mathcal{D}_2(x) > \mathcal{D}_2(1) = \frac{2}{\lambda_0^2} (\lambda_0^2 - \frac{1}{2})^2 > 0.$$

On the other hand (31) applies for  $n \geq 2$  and we see that the orthonormal polynomials satisfy

$$\inf \{ P_n^2(x) + P_{n-1}^2(x) \mid x \in \mathbb{R}, n \in \mathbb{N} \} \geq \frac{2}{\lambda_0^2} (\lambda_0^2 - \frac{1}{2})^2.$$



## 4 Continuous $q$ -ultraspherical polynomials

The continuous  $q$ -ultraspherical polynomials  $C_n(x; \beta|q)$  depend on two real parameters  $q, \beta$ , and for  $|q|, |\beta| < 1$  they are orthogonal with respect to a continuous weight function on  $] -1, 1[$ , cf. [6],[7]. The 3-term recurrence relation is

$$xC_n(x; \beta|q) = \frac{1 - q^{n+1}}{2(1 - \beta q^n)} C_{n+1}(x; \beta|q) + \frac{1 - \beta^2 q^{n-1}}{2(1 - \beta q^n)} C_{n-1}(x; \beta|q), \quad n \geq 0 \quad (34)$$

with  $C_{-1} = 0, C_0 = 1$ . The orthonormal version  $\mathcal{C}_n(x; \beta|q)$  satisfy equation (27) with

$$\lambda_n = \frac{1}{2} \sqrt{\frac{(1 - q^{n+1})(1 - \beta^2 q^n)}{(1 - \beta q^n)(1 - \beta q^{n+1})}}. \quad (35)$$

The value  $C_n(1; \beta|q)$  is not explicitly known, and therefore we can only obtain the recurrence coefficients  $\alpha_n, \gamma_n$  from (1) for  $p_n(x) = C_n(x; \beta|q)/C_n(1; \beta|q)$  as given by the recursive equations

$$\alpha_{n+1} = \frac{\lambda_n^2}{1 - \alpha_n}, \quad \alpha_0 = 0, \quad \gamma_n = 1 - \alpha_n, \quad (36)$$

which we get from the relation  $\lambda_n = \sqrt{\alpha_{n+1}\gamma_n}$ .

**Theorem 4.1.** (i) Assume  $0 \leq \beta \leq q < 1$ . Then the recurrence coefficients  $(\lambda_n)$  form an increasing sequence with limit  $1/2$ , and therefore  $(\mathcal{C}_n(x; \beta|q))$  satisfies (LB).

(ii) Assume  $0 \leq q \leq \beta < 1$ . Then the recurrence coefficients  $(\lambda_n)$  form a decreasing sequence with limit  $1/2$ , and the sequence  $(\alpha_n)$  is increasing and concave with limit  $1/2$ . In particular, we have

$$\frac{\Delta_n(x)}{1 - x^2} \geq c \Delta_n(0), \quad -1 < x < 1, n \geq 1,$$

with  $c = 2\alpha_1(1 - \alpha_2)/(1 - \alpha_1)$ .

*Proof.* The function

$$\psi(x) = \frac{(1 - qx)(1 - \beta^2 x)}{(1 - \beta x)(1 - \beta qx)} = 1 + (1 - \beta)(\beta - q) \frac{x}{(1 - \beta x)(1 - \beta qx)}$$

is decreasing for  $0 \leq \beta \leq q < 1$  and increasing for  $0 \leq q \leq \beta < 1$ . This shows that  $\lambda_n = (1/2)\sqrt{\psi(q^n)}$  is increasing in case (i) and decreasing in case (ii). In both cases the limit is  $1/2$ .

In case (ii) we therefore have  $\lambda_n^2 \geq 1/4$  and hence

$$\alpha_{n+1} \geq \frac{1}{4(1 - \alpha_n)} \geq \alpha_n,$$

because  $4x(1 - x) \leq 1$  for  $0 \leq x \leq 1$ . This shows that  $(\alpha_n)$  is increasing and hence with limit  $1/2$ . We further have

$$\alpha_{n+1} - \alpha_n = 2(\lambda_n^2 - \frac{1}{4}) + 2(\frac{1}{2} - \alpha_n)(\frac{1}{2} - \alpha_{n+1}),$$

which shows that  $\alpha_{n+1} - \alpha_n$  is decreasing, i.e.  $(\alpha_n)$  is concave. We can now apply Theorem 2.5.  $\square$

## References

- [1] H. Alzer, S. Gerhold, M. Kauers, A. Lupaş, On Turán's inequality for Legendre polynomials, *Expo. Math.* **25** no.2 (2007), 181–186.
- [2] Askey, R.: Linearization of the product of orthogonal polynomials, *Problems in Analysis*, R. Gunning, ed., Princeton University Press, Princeton, N.J.(1970), 223–228 .
- [3] A.E. Danese, Explicit evaluations of Turán expressions, *Annali di Matematica Pura ed Applicata*, Serie IV vol. **38** (1955), 339–348.
- [4] J. Dombrowski, P. Nevai, Orthogonal polynomials, measures and recurrence relations. *SIAM J. Math. Anal.* **17** (1986), 752–759.
- [5] F. Filbir, R. Lasser, R. Szwarc, Reiter's condition  $P_1$  and approximate identities for polynomial hypergroups. *Monatsh. Math.* **143** (2004), 189–203. DOI 10.1007/s00605-004-0252-5.
- [6] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Cambridge 2005.
- [7] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report no. 98-17, TU-Delft, 1998.

- [8] A. Máté, P. G. Nevai, Orthogonal polynomials and absolutely continuous measures, In C. K. Chui et. al., editor, Approximation Theory IV, Academic Press, New York, 1983, 611–617.
- [9] P. G. Nevai, Orthogonal polynomials, *Memoirs Amer. Math. Soc.* **18** No. 213 (1979).
- [10] G. Szegő, Orthogonal Polynomials, 4th ed., Colloquium Publications, vol. 23, Amer. Math. Soc., Rhode Island, 1975.
- [11] R. Szwarc, Positivity of Turán determinants for orthogonal polynomials. In: Harmonic Analysis and Hypergroups, Eds. K.A. Ross et al., Birkhäuser, Boston-Basel-Berlin, 1998, 165–182.
- [12] V.R. Thiruvengatchar, T.S. Nannundiah, Inequalities concerning Bessel functions and orthogonal polynomials. *Proc. Indian Acad. Sciences, Section A*, **33** (1951), 373–384.
- [13] P. Turán, On the zeros of the polynomials of Legendre, *Časopis Pest. Mat. Fys.* **75** (1950), 113–122.
- [14] K. Venkatachaliengar, S.K. Lakshmana Rao, On Turán’s Inequality for Ultra-Spherical Polynomials, *Proc. Amer. Math. Soc.* **8** no. 6 (1957), 1075–1087.

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